

# Contents

|   |      |
|---|------|
| <i>Preface</i>  | xi   |
| <i>Acknowledgments</i>  | xiii |
| 1 Direct Solution Methods   | 1    |
| 1.1 Introduction: Networks and Structures   | 2    |
| 1.2 Gaussian Elimination and Matrix Factorization                                       | 7    |
| 1.3 Range and Nullspace   | 14   |
| 1.4 Practical Considerations  | 18   |
| 1.5 Solution of Tridiagonal Systems of Equations  | 24   |
| Exercises   | 30   |
| References  | 44   |
| 2 Theory of Matrix Eigenvalues  | 46   |
| 2.1 The Minimal Polynomial  | 47   |
| 2.2 Selfadjoint and Unitary Matrices  | 54   |
| 2.3 Matrix Equivalence (Similarity Transformations)                                     | 56   |
| 2.4* Normal and $H$ -Normal Matrices  | 62   |
| Exercises   | 66   |
| References  | 82   |
| 3 Positive Definite Matrices, Schur Complements, and<br>Generalized Eigenvalue Problems | 84   |
| 3.1 Positive Definite Matrices  | 85   |
| 3.2 Schur Complements   | 92   |

|      |   |     |
|------|---|-----|
| 3.3  | Condition Numbers   | 95  |
| 3.4* | Estimates of Eigenvalues of Generalized Eigenvalue Problems                                 | 98  |
| 3.5  | Congruence Transformations  | 104 |
| 3.6  | Quasisymmetric Matrices   | 106 |
|      | Exercises   | 107 |
|      | References  | 120 |
| 4    | Reducible and Irreducible Matrices and the Perron-Frobenius Theory for Nonnegative Matrices | 122 |
| 4.1  | Reducible and Irreducible Matrices  | 123 |
| 4.2  | Gershgorin Type Eigenvalue Estimates  | 127 |
| 4.3  | The Perron-Frobenius Theorem  | 134 |
| 4.4  | Rayleigh Quotient and Numerical Range   | 136 |
| 4.5* | Some Estimates of the Perron-Frobenius Root of Nonnegative Matrices                         | 141 |
| 4.6  | A Leontief Closed Input-Output Model  | 146 |
|      | Exercises   | 147 |
|      | References  | 156 |
| 5    | Basic Iterative Methods and Their Rates of Convergence                                      | 158 |
| 5.1  | Basic Iterative Methods   | 160 |
| 5.2  | Stationary Iterative Methods  | 168 |
| 5.3  | The Chebyshev Iterative Method  | 179 |
| 5.4* | The Chebyshev Iterative Method for Matrices with Special Eigenvalue Distributions           | 185 |
|      | Exercises   | 193 |
|      | References  | 198 |
| 6    | $M$ -Matrices, Convergent Splittings, and the SOR Method                                    | 200 |
| 6.1  | $M$ -Matrices   | 202 |
| 6.2  | Convergent Splittings   | 213 |
| 6.3* | Comparison Theorems   | 215 |
| 6.4* | Diagonally Compensated Reduction of Positive Matrix Entries                                 | 222 |

|      |   |     |
|------|---|-----|
| 6.5  | The SOR Method  | 230 |
|      | Exercises   | 245 |
|      | References  | 249 |
| 7    | Incomplete Factorization Preconditioning Methods                                    | 252 |
| 7.1  | Point Incomplete Factorization  | 254 |
| 7.2  | Block Incomplete Factorization; Introduction  | 260 |
| 7.3  | Block Incomplete Factorization of $M$ -Matrices                                     | 263 |
| 7.4  | Block Incomplete Factorization of Positive Definite Matrices                        | 265 |
| 7.5* | Incomplete Factorization Methods for Block $H$ -Matrices                            | 269 |
| 7.6  | Inverse Free Form for Block Tridiagonal Matrices                                    | 281 |
| 7.7  | Symmetrization of Preconditioners and the SSOR and ADI Methods                      | 287 |
|      | Exercises   | 302 |
|      | References  | 310 |
| 8    | Approximate Matrix Inverses and Corresponding Preconditioning Methods               | 314 |
| 8.1  | Two Methods of Computing Approximate Inverses of Block Bandmatrices                 | 315 |
| 8.2  | A Class of Methods for Computing Approximate Inverses of Matrices                   | 321 |
| 8.3  | A Symmetric and Positive Definite Approximate Inverse                               | 335 |
| 8.4  | Combinations of Explicit and Implicit Methods                                       | 342 |
| 8.5  | Methods of Matrix Action  | 350 |
| 8.6  | Decay Rates of (Block-) Entries of Inverses of (Block-) Tridiagonal s.p.d. Matrices | 356 |
|      | References  | 370 |
| 9    | Block Diagonal and Schur Complement Preconditionings                                | 372 |
| 9.1  | The C.B.S. Constant   | 374 |
| 9.2  | Block-Diagonal Preconditioning  | 378 |
| 9.3  | Schur Complement Preconditioning  | 381 |

|      |  |     |
|------|--|-----|
| 9.4  | Full Block-Matrix Factorization Methods  | 383 |
| 9.5  | Indefinite Systems   | 394 |
|      | References   | 400 |
| 10   | Estimates of Eigenvalues and Condition Numbers for<br>Preconditioned Matrices      | 402 |
| 10.1 | Upper Eigenvalue Bounds  | 403 |
| 10.2 | Perturbation Methods   | 413 |
| 10.3 | Lower Eigenvalue Bounds for $M$ -Matrices  | 418 |
| 10.4 | Upper and Lower Bounds of Condition Numbers  | 425 |
| 10.5 | Asymptotic Estimates of Condition Numbers for<br>Second-Order Elliptic Problems    | 433 |
|      | References   | 447 |
| 11   | Conjugate Gradient and Lanczos-Type Methods  | 449 |
| 11.1 | The Three-Term Recurrence Form of the Conjugate<br>Gradient Method                 | 452 |
| 11.2 | The Standard Conjugate Gradient Method   | 459 |
| 11.3 | The Lanczos Method for Generating $A$ -Orthogonal<br>Vectors                       | 480 |
|      | References   | 501 |
| 12   | Generalized Conjugate Gradient Methods   | 504 |
| 12.1 | Generalized Conjugate Gradient, Least Squares Methods                              | 506 |
| 12.2 | Orthogonal Error Methods   | 530 |
| 12.3 | Generalized Conjugate Gradient Methods and Variable<br>(Nonlinear) Preconditioners | 542 |
|      | References   | 555 |
| 13   | The Rate of Convergence of the Conjugate Gradient Method                           | 558 |
| 13.1 | Rate of Convergence Estimates Based on Min Max<br>Approximations                   | 559 |
| 13.2 | Estimates Based on the Condition Number  | 566 |
| 13.3 | An Estimate Based on a Ratio Involving the Trace<br>and the Determinant            | 575 |

|      |  |     |
|------|--|-----|
| 13.4 | Estimates of the Rate of Convergence Using Different Norms | 585 |
| 13.5 | Conclusions  | 591 |
|      | References   | 593 |

Appendices:

|     |   |     |
|-----|---|-----|
| A   | Matrix Norms, Inherent Errors, and Computation of Eigenvalues | 595 |
| A.1 | Vector and Matrix Norms                                       | 596 |
| A.2 | Inherent Errors in Systems of Linear Algebraic Equations      | 605 |
| A.3 | Estimation and Computation of Eigenvalues                     | 609 |
|     | Exercises   | 625 |
|     | References  | 637 |
| B   | Chebyshev Polynomials   | 639 |
|     | References  | 641 |
| C   | Some Inequalities for Functions of Matrices                   | 642 |
| C.1 | Convex Functions  | 644 |
| C.2 | Matrix-Convex Functions                                       | 645 |
|     | References  | 648 |
|     | <i>Index</i>  | 649 |