

# Contents

<b>Series Preface</b>	<b>vii</b>
<b>Preface</b>	<b>ix</b>
<b>Contents of Part 2: Conservation Laws and Elliptic Equations</b>	<b>xvii</b>
<b>0 Prelude</b>	<b>1</b>
<b>1 Introduction to Finite Differences</b>	<b>5</b>
1.1 Introduction . . . . .	5
1.2 Getting Started . . . . .	6
1.2.1 Implementation . . . . .	9
1.3 Consistency . . . . .	10
1.3.1 Special Choice of $\Delta x$ and $\Delta t$ . . . . .	13
1.4 Neumann Boundary Conditions . . . . .	14
1.5 Some Variations . . . . .	16
1.5.1 Lower Order Terms . . . . .	16
1.5.2 Nonhomogeneous Equations and Boundary Conditions . . . . .	17
1.5.3 A Higher Order Scheme . . . . .	18
1.6 Derivation of Difference Equations . . . . .	21
1.6.1 Neumann Boundary Conditions . . . . .	27
1.6.2 Cell Averaged Equations . . . . .	29

1.6.3	Cell Centered Grids . . . . .	32
1.6.4	Nonuniform Grids . . . . .	34
<b>2</b>	<b>Some Theoretical Considerations</b>	<b>41</b>
2.1	Introduction . . . . .	41
2.2	Convergence . . . . .	41
2.2.1	Initial-Value Problems . . . . .	41
2.2.2	Initial-Boundary-Value Problems . . . . .	45
2.2.3	A Review of Linear Algebra . . . . .	48
2.2.4	Some Additional Convergence Topics . . . . .	54
2.3	Consistency . . . . .	55
2.3.1	Initial-Value Problems . . . . .	55
2.3.2	Initial-Boundary-Value Problems . . . . .	65
2.4	Stability . . . . .	73
2.4.1	Initial-Value Problems . . . . .	73
2.4.2	Initial-Boundary-Value Problems . . . . .	77
2.5	The Lax Theorem . . . . .	79
2.5.1	Initial-Value Problems . . . . .	79
2.5.2	Initial-Boundary-Value Problems . . . . .	81
2.6	Computational Interlude I . . . . .	83
2.6.1	Review of Computational Results . . . . .	83
2.6.2	HW0.0.1 . . . . .	84
2.6.3	Implicit Schemes . . . . .	84
2.6.4	Neumann Boundary Conditions . . . . .	89
2.6.5	Derivation of Implicit Schemes . . . . .	93
<b>3</b>	<b>Stability</b>	<b>97</b>
3.1	Analysis of Stability . . . . .	97
3.1.1	Initial-Value Problems . . . . .	97
3.1.2	Initial-Boundary-Value Problems . . . . .	112
3.2	Finite Fourier Series and Stability . . . . .	117
3.3	Gerschgorin Circle Theorem . . . . .	132
3.4	Computational Interlude II . . . . .	137
3.4.1	Review of Computational Results . . . . .	137
3.4.2	HW0.0.1 . . . . .	139
<b>4</b>	<b>Parabolic Equations</b>	<b>147</b>
4.1	Introduction . . . . .	147
4.2	Two Dimensional Parabolic Equations . . . . .	148
4.2.1	Neumann Boundary Conditions . . . . .	151
4.2.2	Derivation of Difference Equations . . . . .	153
4.3	Convergence, Consistency, Stability . . . . .	156
4.3.1	Stability of Initial-Value Schemes . . . . .	157
4.3.2	Stability of Initial-Boundary-Value Schemes . . . . .	160
4.4	Alternating Direction Implicit Schemes . . . . .	164

4.4.1	Peaceman-Rachford Scheme . . . . .	165
4.4.2	Initial-Value Problems . . . . .	165
4.4.3	Initial-Boundary-Value Problems . . . . .	169
4.4.4	Douglas-Rachford Scheme . . . . .	183
4.4.5	Nonhomogeneous ADI Schemes . . . . .	185
4.4.6	Three Dimensional Schemes . . . . .	190
4.5	Polar Coordinates . . . . .	193
<b>5</b>	<b>Hyperbolic Equations</b>	<b>205</b>
5.1	Introduction . . . . .	205
5.2	Initial-Value Problems . . . . .	206
5.3	Numerical Solution of Initial-Value Problems . . . . .	209
5.3.1	One Sided Schemes . . . . .	210
5.3.2	Centered Scheme . . . . .	213
5.3.3	Lax-Wendroff Scheme . . . . .	213
5.3.4	More Explicit Schemes . . . . .	216
5.4	Implicit Schemes . . . . .	216
5.4.1	One Sided Schemes . . . . .	216
5.4.2	Centered Scheme . . . . .	217
5.4.3	Lax-Wendroff Scheme . . . . .	218
5.4.4	Crank-Nicolson Scheme . . . . .	219
5.5	Initial-Boundary-Value Problems . . . . .	220
5.5.1	Periodic Boundary Conditions . . . . .	221
5.5.2	Dirichlet Boundary Conditions . . . . .	221
5.6	Numerical Solution of Initial-Boundary-Value Problems . . . . .	223
5.6.1	Periodic Boundary Conditions . . . . .	223
5.6.2	Dirichlet Boundary Conditions . . . . .	229
5.7	The Courant-Friedrichs-Lewy Condition . . . . .	232
5.8	Two Dimensional Hyperbolic Equations . . . . .	239
5.8.1	Conservation Law Derivation . . . . .	239
5.8.2	Initial-Value Problems . . . . .	242
5.8.3	ADI Schemes . . . . .	247
5.8.4	Courant-Friedrichs-Lewy Condition for Two Dimensional Problems . . . . .	249
5.8.5	Two Dimensional Initial-Boundary-Value Problems . . . . .	251
5.9	Computational Interlude III . . . . .	254
5.9.1	Review of Computational Results . . . . .	254
5.9.2	Convection-Diffusion Equations . . . . .	256
5.9.3	HW0.0.1 . . . . .	257
5.9.4	HW0.0.2 . . . . .	258
<b>6</b>	<b>Systems of Partial Differential Equations</b>	<b>261</b>
6.1	Introduction . . . . .	261
6.2	Initial-Value Difference Schemes . . . . .	263

6.2.1	Flux Splitting . . . . .	278
6.2.2	Implicit Schemes . . . . .	280
6.3	Initial–Boundary–Value Problems . . . . .	284
6.3.1	Boundary Conditions . . . . .	285
6.3.2	Implementation . . . . .	292
6.4	Multilevel Schemes . . . . .	300
6.4.1	Scalar Multilevel Schemes . . . . .	300
6.4.2	Implementation of Scalar Multilevel Schemes . . . . .	306
6.4.3	Multilevel Systems . . . . .	310
6.5	Higher Order Hyperbolic Equations . . . . .	312
6.5.1	Initial–Value Problems . . . . .	314
6.5.2	More . . . . .	320
6.6	Courant–Friedrichs–Lewy Condition for Systems . . . . .	322
6.7	Two Dimensional Systems . . . . .	324
6.7.1	Initial–Value Problems . . . . .	325
6.7.2	Boundary Conditions . . . . .	333
6.7.3	Two Dimensional Multilevel Schemes . . . . .	338
6.8	Simultaneously Diagonalizable Matrices? . . . . .	340
6.9	A Consistent, Convergent, Unstable Difference Scheme? . . . . .	342
6.10	Computational Interlude IV . . . . .	344
6.10.1	HW0.0.1 and HW0.0.2 . . . . .	344
6.10.2	HW0.0.3 . . . . .	347
6.10.3	Parabolic Problems in Polar Coordinates . . . . .	351
6.10.4	An Alternate Scheme for Polar Coordinates . . . . .	356
<b>7</b>	<b>Dispersion and Dissipation</b>	<b>361</b>
7.1	Introduction . . . . .	361
7.1.1	HW5.6.3 . . . . .	361
7.1.2	HW5.6.5 . . . . .	365
7.2	Dispersion and Dissipation for Partial Differential Equations . . . . .	365
7.3	Dispersion and Dissipation for Difference Equations . . . . .	369
7.4	Dispersion Analysis for the Leapfrog Scheme . . . . .	388
7.5	More Dissipation . . . . .	396
7.6	Artificial Dissipation . . . . .	399
7.7	Modified Partial Differential Equation . . . . .	404
7.8	Discontinuous Solutions . . . . .	413
7.9	Computational Interlude V . . . . .	422
7.9.1	HW0.0.1 . . . . .	423
7.9.2	HW0.0.3 . . . . .	425
	<b>References</b>	<b>427</b>
	<b>Index</b>	<b>429</b>