

Contents

	Introduction	1
1.	Partial Differentiation	3
	The concept of partial derivatives. The total differential or change of a dependent function which results from infinitesimal changes of the independent argument variables. Transformation of variables. Relations between partial derivatives of a function with respect to an original and transformed set of independent variables. Conversion of a PDE following a change of independent variables. Composite differentiation.	
2.	Solutions of PDE's and their Specification	17
	Identification of a particular solution of linear PDE's which involve two independent variables through the assignment of data. Solution of a first order equation with assigned values along a plane curve. Solution of a second order equation whose values and those of its normal derivative are separately prescribed along a curve. Characteristics and normal forms of the second order PDE's. Special feature relating to data on characteristic curves.	
3.	PDE's and related Arbitrary Functions	31
	Occurrence of arbitrary functions in the solutions of PDE's. PDE's derived on elimination of arbitrary functions from implicit or explicit relations which involve independent and dependent variables. PDE's descriptive of surfaces of revolution. The PDE, containing an arbitrary function, which identifies a family of parallel surfaces or plane curves. Complete integrals.	

4.	Particular Solutions of PDE's	39
	Solutions with a product form whose factors are functions of a single independent variable each and satisfy ODE's. Solutions which involve a separation constant or parametric variable. A solution represented by a function whose argument is a specific combination of the independent variables. Association of PDE's with continuum or macroscopic models of a physical nature. Form invariance and coordinate transformation. The Laplacian differential operator and spatial symmetry.	
5.	Similarity Solutions	51
	Solutions based on dimensionless combinations of variables. A constructive role for special transformations of the independent/dependent variables which preserve form invariance of the PDE. Conversion of a PDE to an ODE after a suitable transformation of variables. Transformation groups. A similarity solution associated with grazing fluid flow past a flat plate.	
6.	Correctly Set Problems	67
	Continuous dependence of a solution on the defining as a criterion for a correctly set or well posed problem. Examples of improperly posed problems.	
7.	Some Preliminary Aspects of Linear First Order PDE's	71
	Equivalence of a general linear first order PDE with two independent variables and a first order ODE along characteristic plane curves or traces. The role of traces in fixing the solution domain. A composite solution with distinct representations on opposite sides of a trace and a discontinuity in derivative at the trace. A transformation of independent variables, linked to the trace family, which removes one of the partial derivative terms in the PDE.	
8.	Linear First Order PDE's with Two Independent Variables	77

Geometric description of solutions in terms of three-dimensional surfaces. Line elements and tangent planes of solution surfaces. Characteristic curves in space and their specification by a system of ODE's with a parametric variable. Illustrative example for comparison of solution procedures. An initial value problem: the selection of a solution surface which contains a given space curve. Joint use of parametric variables for the characteristic curves and the curve with assigned data. An example involving a non-homogeneous PDE. General features of the trio of ODE's with a parametric independent variable which describe the characteristic curves.

9. First Order Nonlinear PDE's 91

A quasi-linear equation suggested by a model of unidirectional traffic flow; contrast with its linear counterpart for unidirectional wave motion. Analysis of a special case, using the characteristic ODE's together with an initial condition. Sensitive dependence of solutions on the initial data. Intersecting traces and an envelope of traces. Non-uniform solution behavior. Integral surfaces of quasi-linear PDE's which pass through a given space curve on which data is assigned. Integration with the help of multipliers. Integral surfaces and tangent planes. Implicit and general solutions of PDE's.

10. Some Technical Problems and Related PDE's 109

Conservation type relations and their conversion to PDE's after postulating a functional dependence of the interrelated descriptive measures for the model. Detailed elaboration for wavetrains with variable wavelength and frequency. Group velocity and energy density specifications. Chromatographic model analysis based on a conservation relation and a rate law for interaction between adsorber and solute.

11. First Order PDE's with Two Independent Variables, General Theory 119

Geometric aspects of solution surfaces in three dimensions: integral plane elements for a PDE and their cone envelope at any point. A surface strip or collection of plane elements. Integral characteristic strip. Five parametric differential equations for the characteristic curves on the integral surface. The Cauchy problem: to determine an integral surface which passes through a pre-selected space curve.

12. First Order PDE's with Multiple Independent Variables 129

A multi-parameter system of ODE's for characteristic curves. Integrals of the system which describe solutions of the PDE. Inhomogeneous PDE's of the quasi-linear category. Euler equation and its solution.

13. Original Details of the Fourier Approach to Boundary Value Problems. 135

Epochal studies of Fourier relating to second order PDE's with auxiliary conditions. Details from his book (1822) which relate to the solution of Laplace's equation within a semi-infinite plane domain, bounded by two parallel lines and a finite normal segment that joins them, given that the solution vanishes along the former and assumes the value unity along the latter; interpretation as a steady temperature distribution over a plate. Formation of a separated variable series solution containing specific trigonometric factors which vanish at the

parallel boundary lines and secure the proper local behavior. Association of a constant (or boundary datum) with a series of trigonometric terms and specification of the coefficients therein. Checks on the solution. Comments on the significance and technical shortcomings of Fourier's analysis.

14. Eigenfunctions and Eigenvalues 143

Eigenfunctions as the non-trivial solutions of a homogeneous system comprising an ODE and a pair of separate boundary or endpoint conditions; eigenvalues as the special parametric values in the ODE for which a non-trivial solution exists. Connection with boundary/initial value problems for PDE's. Specific examples involving a linear PDE for heat transfer or diffusion, which are distinguished by different boundary conditions. An integral property of the eigenfunctions, namely that the product of any two distinct ones has a vanishing integral over the range between the boundary points. Eigenfunction expansions dictated by given initial conditions. Coefficient determinations in such expansions via the integral property of the eigenfunction basis. The possibility of both positive and negative eigenvalues and its interpretation. Graphical consideration of eigenvalue equations.

15. Eigenfunctions and Eigenvalues, continued 161

Periodic boundary conditions and the occurrence of linearly independent eigenfunctions associated with a common eigenvalue. Coefficient determinations in a series which contains distinct sets of eigenfunctions. A modified diffusion PDE together with a parameter dependent boundary condition; eigenvalue dependence on the parameters. Physical interpretation of the latter feature.

16. Non-Orthogonal Eigenfunctions 169

An example wherein different members of an eigenfunction set do not have a vanishing integral: the novelty, a boundary condition that involves first order partial derivatives with respect to the independent variables separately. A procedure for determining the individual coefficients in an expansion based on such eigenfunctions.

17. Further Example of Fourier Style Analysis 175

Solution of Laplace's PDE inside a circular domain, given a general distribution of boundary values. Use of polar coordinates along with the requirements of single-valuedness and regularity for the establishment of an eigenfunction basis. Summation of the eigenfunction series and presentation of the solution in the form of a single definite integral. Analysis of a boundary/initial value problem for a coupled pair of first order PDE's.

18. Inhomogeneous Problems 183

An inhomogeneous ODE, with a coefficient parameter, and a pair of boundary conditions (one inhomogeneous) at two distinct points; the solution expressed as a series of related eigenfunctions or non-trivial solutions of the corresponding homogeneous system. Verification of the solution. Solution of a system, comprising a PDE of diffusion type, two inhomogeneous boundary conditions and an initial condition, by resolution into component parts, one of which evidently complies with the inhomogeneous conditions while the other follows from a straightforward Fourier analysis. An example in which the inhomogeneous PDE admits a particular solution that facilitates construction of the complete solution. A general procedure for directly resolving systems which may possess an inhomogeneous aspect in both the PDE and the boundary conditions by expansion in terms of the related eigenfunctions. Coefficient functions in the expansion obtained from individual ODE's and an initial condition. Detailed examples and remarks on asymptotic (or long time) behavior of the solution.

19. Local Heat Sources 203

Description of a local source by a temperature distribution that is continuous everywhere and has a discontinuous first derivative which implies a net flux of heat away from the source point. Eigenfunction expansions for the temperature on opposite sides of a steady source. Asymptotic form of solution derived from a system of ODE's and consistency check. General time varying source located in a closed circular ring and an instantaneously acting source.

20. An Inhomogeneous Configuration 213

Unidimensional diffusion in a composite setting formed by two sections which have distinct (through uniform) thermal parameters and lengths. Appropriate solutions of the PDE's in the respective sections and their required matching at the point of contact. Eigenfunctions and their role in securing an initial condition.

21. Other Eigenfunction/Eigenvalue Problems 221

A PDE relating to elastic waves and the occurrence of complex eigenvalues for particular boundary conditions; properties of the eigenfunctions. A modified second order wave equation, which a mixed partial derivative term, and complex-valued eigenfunctions. The use of an eigenfunction series to accomodate a pair of initial conditions. Eigenfunctions and eigenvalues of a fourth order PDE.

22. Uniqueness of Solutions 241

Analytic demonstration that a single solution of the homogeneous PDE for diffusion is pinpointed after one initial and a pair of boundary conditions are given. The Maximum Principle for continuous (sourceless) solutions of the diffusion equation within a plane rectangular domain defined by pairs of fixed values for the independent variables. Properties of solutions to the PDE's for diffusion and wave motion arrived at by using domain and curvilinear integrals in combination with differential identities or conservation relations.

23. Alternative Representations of Solutions 253

Different forms of the solution to a well posed problem, which possesses a unique solution, are possible; demonstration for a system involving the diffusion PDE in which the respective series solutions are rapidly convergent at large/small values of the time, and thus have a complementary usefulness.

24. Other differential equations and Inferences Therefrom 269

A PDE for diffusion in the circumstance of position dependent thermal and material parameters; and the related class of Sturm-Liouville eigenfunctions which obey variable coefficient ODE's. PDE's for mass flow and small amplitude acoustical excitations in a horn with non-uniform cross section. Appearance of variable coefficients after transformation of PDE's to exploit suitable coordinate system.

25. Second Order ODE's 285

Different versions of linear, second order homogeneous ODE's and a normal form. Solutions made precise by initial conditions and zeros of same. Linearly independent solutions with interlacing or alternating zeros. Features of solutions derived from a pair of ODE's, one termed a comparison equation and distinguished by a different coefficient function. The consequences, as regards location and number of zeros, which reflect variations in the parametric argument variable of a coefficient function in the ODE. Argument and phase functions in representations of solutions and their role in enforcing prescribed boundary conditions. Consideration of non-homogeneous systems of ODE's and boundary conditions.

26. Boundary Value Problems and Sturm-Liouville Theory 307

Theory pertaining to linear systems which comprise an n^{th} order ODE and an equal number of forms that involve the considered function and its derivatives at the endpoints of the chosen interval; compatibility and the inclusion of a parametric variable. Detailed analysis based on a joint role for the differential operator present in the ODE and an adjoint operator. The linear homogeneous system and its adjoint;

requirements for self-adjointness. Eigenfunctions and eigenvalues of a homogeneous system with a parameter. Self-adjoint systems and orthogonal eigenfunctions. Eigenvalue properties. Discussion of a second order system, employing different representations of the boundary conditions.

27. Green's Functions and Boundary Value Problems 331

Nature of a Green's function for a self-adjoint linear second order ODE: a function with two argument variables whose regular behavior ceases to hold at a single point where the first derivative has a discontinuity and the argument variables are equal. Symmetry of the Green's function with respect to an exchange of its argument variables. The representation of a particular Green's function suggested by the PDE for diffusion along with a pair of homogeneous boundary conditions. Solution of an inhomogeneous ODE expressed in an integral form by means of a Green's function. An integral equation reformulation of a homogeneous boundary value problem which relies on a Green's function. Association of the integral equation with eigenfunctions and eigenvalues. Employment of a Green's function for resolving inhomogeneous boundary value problems. A multidimensional Green's function which enables the solution of the Laplace PDE within a rectangular domain, given an arbitrary assignment of values along the boundary.

28. Green's Functions and Generalizations 361

Non-uniqueness in the solution of a system defined by an inhomogeneous ODE and a pair of homogeneous boundary conditions when a non-trivial solution of the related homogeneous system exists. Definition and construction of a modified Green's function in such circumstances: an example wherein the Green's function is symmetric with respect to the argument variables. Green's functions containing a parameter and their bilinear expansion in terms of related eigenfunctions and eigenvalues; the occurrence of a singular term in the expansion whenever the parameter coincides with an eigenvalue. The adjoint Green's function and its analogous expansion. Common features of adjoint equations and the Green's function representation based on solutions to homogeneous systems and their adjoints.

29. PDE's, Green's Functions and Integral Equations 401

A PDE for transverse vibrations of a string with fixed endpoints which are caused by an external applied force that varies periodically in time and arbitrarily along its length; replacement of the equations for the synchronous motion by a single integral equation after introducing a suitable Green's function. A manner of solving this type of integral equation with supporting details. Specific aspects of the solution for

the string problem and comments on a resonant behavior. An inhomogeneous PDE of diffusion type with an extended source term that has a periodic time dependence; an integral equation reformulation and verification that the solution of the problem is unique, whatever the source frequency. General results which follow from an integral equation restatement of a problem pertaining to a system that comprises an inhomogeneous ODE and a pair of homogeneous boundary conditions.

30. Singular and Infinite Range Problems 425

Boundary value problems characterized by infinite intervals and/or coefficient functions (of the highest order derivative in the ODE) which vanish at an endpoint. Contrasting features of two (single variable) Green's functions: one, defined by a constant coefficient differential operator, which has pole type singularities relative to its parametric variable and another operator, wherein the coefficient of the highest order (second) derivative vanishes at an endpoint, which exhibits a corresponding singularity of the branch line/cut nature. Combination of two Green's functions to express another associated with a PDE and alternative representations of same. Limiting behavior of a Green's function when the original and finite domain becomes a semi-infinite one; the corresponding replacement of a discrete eigenvalue spectrum by a continuous one. Proper boundary conditions at infinity. The linear oscillator in quantum mechanics; an example which reveals the existence of a discrete eigenvalue spectrum although the range of the independent variable is an infinite one.

31. Orthogonality and its Ramifications 457

Coefficient determination in an interpolation formula, based on a fit at equidistant points with a trigonometric polynomial; limiting form of the coefficients when the degree of the polynomial becomes infinite and a trigonometric series representation is indicated. Orthogonal and orthonormal sets of functions. Representation of square integrable (or L_2) functions by an orthonormal series; convergence in the mean. Mean square approximation to L_2 functions. The Bessel inequality and Parseval's equation. Complete orthonormal sets.

32. Fourier Expansions: Generalities 483

Series with variable terms: uniform convergence and a related comparison test. Termwise integration and differentiation of trigonometric (or Fourier) series. Non-uniform convergence and the Gibbs phenomena. An initial/boundary value problem for the diffusion type PDE and verification that the resultant series solution conforms with the inhomogeneous boundary conditions. Partial sums of Fourier series and convergence criteria.

33. Fourier Expansions: Varied Examples 515

Details of the Fourier series which have their origins in different problems of analysis, geometry or technical modelling; comments on particular features of the individual series representations.

34. Fourier Integrals and Transforms 555

The transition from a Fourier series expansion for a function defined on a finite interval to a Fourier integral representation for a function defined over an infinite interval. Complex version of a Fourier integral and the Fourier transform of an integrable function. Fourier integral representations with even/odd symmetry. A relation for square integrable functions and their Fourier transforms. Application of Fourier integrals in the resolution of the diffusion PDE over an unbounded interval. Fourier integral analysis pertaining to a modified diffusion equation that describes the concentration of solute in a flow.

35. Applications of Fourier Transforms 577

Solution of the Laplace PDE within an unbounded strip domain, given an assignment of values on the parallel boundary lines and a behavior at infinity; likewise, within a quadrant of the plane. Construction and discussion of an infinite domain, one-dimensional, Green's function of the diffusion PDE. Analysis of radial diffusion outside a spherical cavity based on a Fourier sine integral representation. The use of multiple Fourier integrals to solve the Poisson (or inhomogeneous Laplace) PDE in three independent variables.

36. Legendre Polynomials and Related Expansions 603

Separated variable solutions of the Laplace PDE which involve functions of the radial coordinate and the (spherical) polar angle coordinate respectively. Discussion of the angle factor ODE and the specification of regular polynomial solutions. The expansion of a fundamental source solution of the Laplace PDE, with its singularity on the polar axis, in terms of Legendre polynomials and powers of the radial coordinate. Orthogonality, normalization and a generating function of the Legendre polynomials. Fourier type expansions based on Legendre polynomials. The representation of a steady temperature distribution throughout a half-space, given its values on a plane boundary surface, by series containing Legendre polynomials.

37. Bessel Functions and Related Expansions 623

The second order (variable) coefficient cylinder function ODE: appearance in separated variable solutions of the Laplace PDE written in terms of cylindrical space coordinates. A PDE for wave motions in

spherical polar coordinates and a class of separable solutions involving cylinder functions. Linearly independent solutions of the cylinder function ODE, the Bessel and Neumann functions; their behaviors for small/large magnitudes of the argument variable. Bessel functions as eigenfunctions of a self-adjoint boundary value problem. Recurrence relations, series expansions and integrals of cylinder functions. Fourier-Bessel series expansions and Fourier-Bessel integral transforms.

38. Hyperbolic Equations 665

Analysis of second order PDE's of hyperbolic nature by means of characteristic curves and variables. The solution of a simple PDE with two independent variables, in accordance with assigned values on a pair of characteristic line segments. Connection between regularity of the assigned data and that of the solution. Domain of dependence of a solution. The one-dimensional PDE of wave type in conjunction with pairs of initial and boundary conditions; contrast between characteristic and separated variable approaches. A Fourier integral type of solution in the circumstance of an unbounded coordinate range. Consideration of a modified wave equation.

Afterwords 699

Bibliography 701

Index 703