

# Contents

<b>Introduction: What Are Partial Differential Equations? . . . . .</b>	<b>1</b>
<b>1. The Laplace Equation as the Prototype of an Elliptic Partial Differential Equation of Second Order . . . . .</b>	<b>7</b>
1.1 Harmonic Functions. Representation Formula for the Solution of the Dirichlet Problem on the Ball (Existence Techniques 0) . . . . .	7
1.2 Mean Value Properties of Harmonic Functions. Subharmonic Functions. The Maximum Principle . . . . .	16
<b>2. The Maximum Principle . . . . .</b>	<b>33</b>
2.1 The Maximum Principle of E. Hopf . . . . .	33
2.2 The Maximum Principle of Alexandrov and Bakelman . . . . .	39
2.3 Maximum Principles for Nonlinear Differential Equations . . . . .	44
<b>3. Existence Techniques I: Methods Based on the Maximum Principle . . . . .</b>	<b>53</b>
3.1 Difference Methods: Discretization of Differential Equations . . . . .	53
3.2 The Perron Method . . . . .	62
3.3 The Alternating Method of H.A. Schwarz . . . . .	66
3.4 Boundary Regularity . . . . .	71
<b>4. Existence Techniques II: Parabolic Methods. The Heat Equation . . . . .</b>	<b>79</b>
4.1 The Heat Equation: Definition and Maximum Principles . . . . .	79
4.2 The Fundamental Solution of the Heat Equation. The Heat Equation and the Laplace Equation . . . . .	91
4.3 The Initial Boundary Value Problem for the Heat Equation . . . . .	98
4.4 Discrete Methods . . . . .	114
<b>5. Reaction-Diffusion Equations and Systems . . . . .</b>	<b>119</b>
5.1 Reaction-Diffusion Equations . . . . .	119
5.2 Reaction-Diffusion Systems . . . . .	126
5.3 The Turing Mechanism . . . . .	130

<b>6. The Wave Equation and its Connections with the Laplace and Heat Equations</b>	139
6.1 The One-Dimensional Wave Equation	139
6.2 The Mean Value Method: Solving the Wave Equation through the Darboux Equation	143
6.3 The Energy Inequality and the Relation with the Heat Equation	147
<b>7. The Heat Equation, Semigroups, and Brownian Motion</b>	153
7.1 Semigroups	153
7.2 Infinitesimal Generators of Semigroups	155
7.3 Brownian Motion	171
<b>8. The Dirichlet Principle. Variational Methods for the Solution of PDEs (Existence Techniques III)</b>	183
8.1 Dirichlet's Principle	183
8.2 The Sobolev Space $W^{1,2}$	186
8.3 Weak Solutions of the Poisson Equation	196
8.4 Quadratic Variational Problems	198
8.5 Abstract Hilbert Space Formulation of the Variational Problem. The Finite Element Method	201
8.6 Convex Variational Problems	209
<b>9. Sobolev Spaces and <math>L^2</math> Regularity Theory</b>	219
9.1 General Sobolev Spaces. Embedding Theorems of Sobolev, Morrey, and John–Nirenberg	219
9.2 $L^2$ -Regularity Theory: Interior Regularity of Weak Solutions of the Poisson Equation	234
9.3 Boundary Regularity and Regularity Results for Solutions of General Linear Elliptic Equations	241
9.4 Extensions of Sobolev Functions and Natural Boundary Conditions	249
9.5 Eigenvalues of Elliptic Operators	255
<b>10. Strong Solutions</b>	271
10.1 The Regularity Theory for Strong Solutions	271
10.2 A Survey of the $L^p$ -Regularity Theory and Applications to Solutions of Semilinear Elliptic Equations	276
<b>11. The Regularity Theory of Schauder and the Continuity Method (Existence Techniques IV)</b>	283
11.1 $C^\alpha$ -Regularity Theory for the Poisson Equation	283
11.2 The Schauder Estimates	293
11.3 Existence Techniques IV: The Continuity Method	299

<b>12. The Moser Iteration Method and the Regularity Theorem of de Giorgi and Nash</b> .....	305
12.1 The Moser–Harnack Inequality .....	305
12.2 Properties of Solutions of Elliptic Equations .....	317
12.3 Regularity of Minimizers of Variational Problems .....	321
<b>Appendix. Banach and Hilbert Spaces. The <math>L^p</math>-Spaces</b> .....	339
<b>References</b> .....	347
<b>Index of Notation</b> .....	349
<b>Index</b> .....	353