

Contents

Acknowledgements	vii
General conventions	xvii
Introduction	xxi
0.1 The subject matter of the present book	xxi
0.1.1 Tool and object	xxi
0.1.2 Stages of development of category theory	xxiv
0.1.3 The plan of the book	xxv
0.1.4 What is not in this book	xxviii
0.2 Secondary literature and sources	xxix
0.2.1 Historical writing on category theory: the state of the art and a necessary change of perspective	xxx
0.2.2 Philosophical writing on CT	xxxix
0.2.3 Unpublished sources	xxxix
0.2.3.1 Bourbaki	xxxix
0.2.3.2 The Samuel Eilenberg records at Columbia Uni- versity. A recently rediscovered collection	xxxix
0.2.4 Interviews with witnesses	xxxix
0.3 Some remarks concerning historical methodology	xxxix
0.3.1 How to find and how to organize historical facts	xxxix
0.3.2 Communities	xxxix
0.3.2.1 What is a community?	xxxix
0.3.2.2 How can one recognize a community?	xxxix
0.3.2.3 “Mainstream” mathematics	xxxix
1 Prelude: Poincaré, Wittgenstein, Peirce, and the use of concepts	1
1.1 A plea for philosophy of mathematics	3
1.1.1 The role of philosophy in historical research, and vice versa	3
1.1.2 The debate on the relevance of research in foundations of mathematics	6
1.2 Using concepts	7

1.2.1	Formal definitions and language games	7
1.2.1.1	Correct use and reasonable use	7
1.2.1.2	The learning of informal application rules	9
1.2.1.3	The interaction between a concept and its intended uses	10
1.2.2	How we make choices	11
1.2.2.1	The term “theory” and the criterion problem . . .	11
1.2.2.2	The task of the philosopher, described by Poincaré and others	13
1.2.2.3	The role of applications	15
1.2.3	Uses as tool and uses as object	16
1.2.3.1	Problem solving, conceptual clarification and “split- ting off”	16
1.2.3.2	Questioning of formerly tacit beliefs	20
1.3	Reductionist vs. pragmatist epistemology of mathematics	21
1.3.1	Criticizing reductionism	22
1.3.1.1	Peirce on reductionism	23
1.3.1.2	Peirce on prejudices, and the history of concepts .	25
1.3.1.3	Wittgenstein’s criticism of reductionism	26
1.3.1.4	Criticizing formalism	29
1.3.2	A new conception of intuition	30
1.3.2.1	Some uses of the term “intuition”	30
1.3.2.2	Intuitive uses and common senses	32
1.3.2.3	Provisional validity	34
1.3.2.4	What is accomplished by this new conception of intuition?	36
1.3.2.5	One more criticism of reductionism	36
1.3.2.6	Counterarguments	37
2	Category theory in Algebraic Topology	39
2.1	Homology theory giving rise to category theory	40
2.1.1	Homology groups before Noether and Vietoris	41
2.1.2	Homology and the study of mappings	41
2.1.2.1	Hopf’s group-theoretical version of Lefschetz’ fixed point formula and the “algebra of mappings” . . .	42
2.1.2.2	Hopf’s account of the $K^n \rightarrow S^n$ problem	44
2.1.2.3	An impulse for Algebra: homomorphisms are not always surjective	45
2.1.2.4	The use of the arrow symbol	46
2.1.3	Homology theory for general spaces	49
2.1.4	The work of Walther Mayer on chain complexes	51
2.2	Eilenberg and Mac Lane: Group extensions and homology	51
2.2.1	The respective works of Eilenberg and Mac Lane giving way to the collaboration	51

2.2.1.1	Eilenberg: the homology of the solenoid	52
2.2.1.2	Mac Lane: group extensions and class field theory	52
2.2.1.3	The order of arguments of the functor Ext	53
2.2.2	The meeting	54
2.2.3	The results of Eilenberg and Mac Lane and universal coefficient theorems	55
2.2.4	Excursus: the problem of universal coefficients	56
2.2.5	Passage to the limit and “naturality”	58
2.2.6	The isomorphism theorem for inverse systems	60
2.3	The first publications on category theory	61
2.3.1	New conceptual ideas in the 1945 paper	61
2.3.1.1	Concepts of category theory and the original context of their introduction	61
2.3.1.2	Functorial treatment of direct and inverse limits	63
2.3.2	The reception of the 1945 paper	65
2.3.2.1	Eilenberg and Mac Lane needed to have courage to write the paper	65
2.3.2.2	Reasons for the neglect: too general or rather not general enough?	66
2.3.3	Reviewing the folklore history	67
2.3.4	Informal parlance	69
2.3.4.1	“Natural transformation”	70
2.3.4.2	“Category”	74
2.4	Eilenberg and Steenrod: Foundations of algebraic topology	76
2.4.1	An axiomatic approach	77
2.4.1.1	The project: axiomatizing “homology theories”	77
2.4.1.2	Axiomatics and exposition	78
2.4.1.3	A theory of theories	80
2.4.2	The significance of category theory for the enterprise	81
2.4.3	Mac Lane’s paper on duality for groups	83
2.5	Simplicial sets and adjoint functors	87
2.5.1	Complete semisimplicial complexes	87
2.5.2	Kan’s conceptual innovations	90
2.6	Why was CT first used in algebraic topology and not elsewhere?	91
3	Category theory in Homological Algebra	93
3.1	Homological algebra for modules	96
3.1.1	Cartan and Eilenberg: derived Functors	96
3.1.1.1	The aims of the 1956 book	96
3.1.1.2	Satellites and derived functors: abandoning an intuitive concept	98
3.1.1.3	The derivation procedure	98
3.1.2	Buchsbaum’s dissertation	100
3.1.2.1	The notion of exact category	100

3.1.2.2	Buchsbaum's achievement: duality	101
3.2	Development of the sheaf concept until 1957	104
3.2.1	Leray: (pre)sheaves as coefficient systems for algebraic topology	106
3.2.1.1	Leray's papers of 1946	106
3.2.1.2	On the reception of these works outside France	107
3.2.2	The "Séminaire Cartan"	109
3.2.2.1	Sheaf theory in two attempts	110
3.2.2.2	The new sheaf definition: "espaces étalés"	111
3.2.2.3	Sheaf cohomology in the Cartan seminar	115
3.2.3	Serre and "Faisceaux algébriques cohérents"	117
3.2.3.1	Sheaf cohomology in Algebraic Geometry?	117
3.2.3.2	Čech cohomology as a substitute for fine sheaves	118
3.2.3.3	The cohomology sequence for coherent sheaves	118
3.3	The Tôhoku paper	119
3.3.1	How the paper was written	120
3.3.1.1	The main source: the Grothendieck–Serre correspondence	120
3.3.1.2	Grothendieck's Kansas travel, and his report on fibre spaces with structure sheaf	125
3.3.1.3	Preparation and publication of the manuscript	127
3.3.2	Grothendieck's work in relation to earlier work in homological algebra	128
3.3.2.1	Grothendieck's awareness of the earlier work	128
3.3.2.2	Grothendieck's adoption of categorical terminology	129
3.3.2.3	The "classe abélienne"-terminology	130
3.3.3	The plan of the Tôhoku paper	131
3.3.3.1	Sheaves are particular functors on the open sets of a topological space	132
3.3.3.2	Sheaves form an abelian category	134
3.3.3.3	The concentration on injective resolutions	135
3.3.3.4	The proof that there are enough injective sheaves	137
3.3.3.5	Furnishing spectral sequences by injective resolutions and the Riemann–Roch–Hirzebruch–Grothendieck theorem	140
3.3.4	Grothendieck's category theory and its job in his proofs	142
3.3.4.1	Basic notions: infinitary arrow language	142
3.3.4.2	"Diagram schemes" and $\text{Open}(X)^{\text{op}}$	147
3.3.4.3	Equivalence of categories and its role in the proof that there are enough injective sheaves	148
3.3.4.4	Diagram chasing and the full embedding theorem	151
3.4	Conclusions	153
3.4.1	Transformation of the notion of homology theory: the accent on the abelian variable	153

3.4.2	Two mostly unrelated communities?	155
3.4.3	Judgements concerning the relevance of Grothendieck's contribution	158
3.4.3.1	Was Grothendieck the founder of category theory as an independent field of research?	158
3.4.3.2	From a language to a tool?	159
4	Category theory in Algebraic Geometry	161
4.1	Conceptual innovations by Grothendieck	163
4.1.1	From the concept of variety to the concept of scheme	163
4.1.1.1	Early approaches in work of Chevalley and Serre	163
4.1.1.2	Grothendieck's conception and the undermining of the "sets with structure" paradigm	164
4.1.1.3	The moduli problem and the notion of representable functor	169
4.1.1.4	The notion of geometrical point and the categorical predicate of having elements	170
4.1.2	From the Zariski topology to Grothendieck topologies	172
4.1.2.1	Problems with the Zariski topology	172
4.1.2.2	The notion of Grothendieck topology	174
4.1.2.3	The topos is more important than the site	175
4.2	The Weil conjectures	178
4.2.1	Weil's original text	178
4.2.2	Grothendieck's reception of the conjectures and the search for the Weil cohomology	181
4.2.3	Grothendieck's visions: Standard conjectures, Motives and Tannaka categories	185
4.3	Grothendieck's methodology and categories	189
5	From tool to object: full-fledged category theory	193
5.1	Some concepts transformed in categorical language	194
5.1.1	Homology	194
5.1.2	Complexes	195
5.1.3	Coefficients for homology and cohomology	196
5.1.4	Sheaves	199
5.2	Important steps in the theory of functors	200
5.2.1	Hom-Functors	200
5.2.2	Functor categories	202
5.2.3	The way to the notion of adjoint functor	202
5.2.3.1	Delay?	203
5.2.3.2	Unresistant examples	204
5.2.3.3	Reception in France	206
5.3	What is the concept of object about?	207
5.3.1	Category theory and structures	207

5.3.1.1	Bourbaki's structuralist ontology	208
5.3.1.2	The term "structure" and Bourbaki's trial of an explication	209
5.3.1.3	The structuralist interpretation of mathematics re- visited	210
5.3.1.4	Category theory and structural mathematics . . .	211
5.3.1.5	Categories of sets with structure—and all the rest	214
5.3.2	The language of arrow composition	218
5.3.2.1	Objects cannot be penetrated	218
5.3.2.2	The criterion of identification for objects: equal up to isomorphism	221
5.3.2.3	The relation of objects and arrows	222
5.3.2.4	Equality of functions and of arrows	223
5.4	Categories as objects of study	223
5.4.1	Category: a generalization of the concept of group? . . .	223
5.4.2	Categories as domains and codomains of functors	224
5.4.3	Categories as graphs	225
5.4.4	Categories as objects of a category?	228
5.4.4.1	Uses of Cat	228
5.4.4.2	The criterion of identification for categories	229
5.4.4.3	Cat is no category	232
6	Categories as sets: problems and solutions	235
6.1	Preliminaries on the problems and their interpretation	237
6.1.1	Naive category theory and its problems	237
6.1.2	Legitimate sets	239
6.1.3	Why aren't we satisfied just with small categories?	241
6.2	Preliminaries on methodology	241
6.2.1	Chronology of problems and solutions	241
6.2.2	The parties of the discussion	243
6.2.3	Solution attempts not discussed in the present book	244
6.3	The problems in the age of Eilenberg and Mac Lane	245
6.3.1	Their description of the problems	245
6.3.2	The fixes they propose	247
6.4	The problems in the era of Grothendieck's Tôhoku paper	248
6.4.1	Hom-sets	248
6.4.2	Mac Lane's first contribution to set-theoretical foundations of category theory	248
6.4.2.1	Mac Lane's contribution in the context of the two disciplines	249
6.4.2.2	Mac Lane's observations	250
6.4.2.3	Mac Lane's fix: locally small categories	251
6.4.3	Mitchell's use of "big abelian groups"	253
6.4.4	The French discussion	253

6.4.4.1	The awareness of the problems	253
6.4.4.2	Grothendieck's fix, and the Bourbaki discussion on set-theoretical foundations of category theory . . .	255
6.4.4.3	Grothendieck universes in the literature: Sonner, Gabriel, and SGA	261
6.4.5	The history of inaccessible cardinals: the roles of Tarski and of category theory	263
6.4.5.1	Inaccessibles before 1938	264
6.4.5.2	Tarski's axiom $\mathcal{A}$ and its relation to Tarski's theory of truth	264
6.4.5.3	A reduction of activity in the field—and a revival due to category theory?	266
6.4.6	Significance of Grothendieck universes as a foundation for category theory	266
6.4.6.1	Bourbaki's "hypothetical-deductive doctrine", and relative consistency of $\mathcal{A}$ with ZF	267
6.4.6.2	Is the axiom of universes adequate for practice of category theory?	269
6.4.6.3	Naive set theory, the "universe of discourse" and the role of large cardinal hypotheses	269
6.5	Ehresmann's fix: allowing for "some" self-containing	273
6.6	Kreisel's fix: how strong a set theory is really needed?	276
6.7	The last word on set-theoretical foundations?	279
7	Categorical foundations	281
7.1	The concept of foundation of mathematics	282
7.1.1	Foundations: mathematical and philosophical	282
7.1.2	Foundation or river bed?	283
7.2	Lawvere's categorical foundations: a historical overview	284
7.2.1	Lawvere's elementary characterization of Set	284
7.2.2	Lawvere's tentative axiomatization of the category of all cat- egories	285
7.2.3	Lawvere on what is universal in mathematics	288
7.3	Elementary toposes and "local foundations"	290
7.3.1	A surprising application of Grothendieck's algebraic geome- try: "geometric logic"	290
7.3.2	Toposes as foundation	291
7.3.2.1	The relation between categorical set theory and ZF	292
7.3.2.2	Does topos theory presuppose set theory?	294
7.4	Categorical foundations and foundational problems of CT	295
7.4.1	Correcting the historical folklore	295
7.4.2	Bénabou's categorical solution for foundational problems of CT	296

7.5	General objections, in particular the argument of “psychological priority”	300
8	Pragmatism and category theory	303
8.1	Category theorists and category theory	303
8.1.1	The implicit philosophy: realism?	303
8.1.2	The common sense of category theorists	306
8.1.3	The intended model: a theory of theories	310
8.2	Which epistemology for mathematics?	314
8.2.1	Reductionism does not work	314
8.2.2	Pragmatism works	315
A	Abbreviations	317
A.1	Bibliographical information and related things	317
A.1.1	General abbreviations	317
A.1.2	Publishers, institutes and research organizations	317
A.1.3	Journals, series	318
A.2	Mathematical symbols and abbreviations	318
A.3	Bourbaki	319
	Bibliography	321
	Indexes	341
	Author index	342
	Subject index	347