
Contents

Preface to the Second Edition	xvii
Preface to the First Edition	xix
0 Preliminaries	1
1 Green's Theorem	1
1.1 Differential Operators and Adjoint	2
2 The Continuity Equation	3
3 The Heat Equation and the Laplace Equation	5
3.1 Variable Coefficients	5
4 A Model for the Vibrating String	6
5 Small Vibrations of a Membrane	8
6 Transmission of Sound Waves	11
7 The Navier–Stokes System	13
8 The Euler Equations	13
9 Isentropic Potential Flows	14
9.1 Steady Potential Isentropic Flows	15
10 Partial Differential Equations	16
1 Quasi-Linear Equations and the Cauchy–Kowalewski	
Theorem	17
1 Quasi-Linear Second-Order Equations in Two Variables	17
2 Characteristics and Singularities	19
2.1 Coefficients Independent of u_x and u_y	20
3 Quasi-Linear Second-Order Equations	21
3.1 Constant Coefficients	23
3.2 Variable Coefficients	23
4 Quasi-Linear Equations of Order $m \geq 1$	24
4.1 Characteristic Surfaces	25
5 Analytic Data and the Cauchy–Kowalewski Theorem	26
5.1 Reduction to Normal Form ([19])	26

6	Proof of the Cauchy–Kowalewski Theorem	27
6.1	Estimating the Derivatives of u at the Origin	28
7	Auxiliary Inequalities	29
8	Auxiliary Estimations at the Origin	31
9	Proof of the Cauchy–Kowalewski Theorem (Concluded)	32
9.1	Proof of Lemma 6.1	33
	Problems and Complements	33
1c	Quasi-Linear Second-Order Equations in Two Variables	33
5c	Analytic Data and the Cauchy–Kowalewski Theorem	34
6c	Proof of the Cauchy–Kowalewski Theorem	34
8c	The Generalized Leibniz Rule	34
9c	Proof of the Cauchy–Kowalewski Theorem (Concluded)	35
2	The Laplace Equation	37
1	Preliminaries	37
1.1	The Dirichlet and Neumann Problems	38
1.2	The Cauchy Problem	39
1.3	Well-Posedness and a Counterexample of Hadamard	39
1.4	Radial Solutions	40
2	The Green and Stokes Identities	41
2.1	The Stokes Identities	41
3	Green’s Function and the Dirichlet Problem for a Ball	43
3.1	Green’s Function for a Ball	45
4	Sub-Harmonic Functions and the Mean Value Property	47
4.1	The Maximum Principle	50
4.2	Structure of Sub-Harmonic Functions	50
5	Estimating Harmonic Functions and Their Derivatives	52
5.1	The Harnack Inequality and the Liouville Theorem	52
5.2	Analyticity of Harmonic Functions	53
6	The Dirichlet Problem	55
7	About the Exterior Sphere Condition	58
7.1	The Case $N = 2$ and ∂E Piecewise Smooth	59
7.2	A Counterexample of Lebesgue for $N = 3$ ([101])	59
8	The Poisson Integral for the Half-Space	60
9	Schauder Estimates of Newtonian Potentials	62
10	Potential Estimates in $L^p(E)$	65
11	Local Solutions	68
11.1	Local Weak Solutions	69
12	Inhomogeneous Problems	70
12.1	On the Notion of Green’s Function	70
12.2	Inhomogeneous Problems	71
12.3	The Case $f \in C_o^\infty(E)$	72
12.4	The Case $f \in C^\eta(\bar{E})$	72

Problems and Complements	73
1c Preliminaries	73
1.1c Newtonian Potentials on Ellipsoids	73
1.2c Invariance Properties	74
2c The Green and Stokes Identities	74
3c Green's Function and the Dirichlet Problem for the Ball	74
3.1c Separation of Variables	75
4c Sub-Harmonic Functions and the Mean Value Property	76
4.1c Reflection and Harmonic Extension	77
4.2c The Weak Maximum Principle	77
4.3c Sub-Harmonic Functions	78
5c Estimating Harmonic Functions	79
5.1c Harnack-Type Estimates	80
5.2c Ill-Posed Problems: An Example of Hadamard	80
5.3c Removable Singularities	81
7c About the Exterior Sphere Condition	82
8c Problems in Unbounded Domains	83
8.1c The Dirichlet Problem Exterior to a Ball	83
9c Schauder Estimates up to the Boundary ([135, 136])	84
10c Potential Estimates in $L^p(E)$	84
10.1c Integrability of Riesz Potentials	85
10.2c Second Derivatives of Potentials	85
3 Boundary Value Problems by Double-Layer Potentials	87
1 The Double-Layer Potential	87
2 On the Integral Defining the Double-Layer Potential	89
3 The Jump Condition of $W(\partial E, x_o; v)$ Across ∂E	91
4 More on the Jump Condition Across ∂E	93
5 The Dirichlet Problem by Integral Equations ([111])	94
6 The Neumann Problem by Integral Equations ([111])	95
7 The Green Function for the Neumann Problem	97
7.1 Finding $\mathcal{G}(\cdot; \cdot)$	98
8 Eigenvalue Problems for the Laplacian	99
8.1 Compact Kernels Generated by Green's Function	100
9 Compactness of A_F in $L^p(E)$ for $1 \leq p \leq \infty$	100
10 Compactness of A_ϕ in $L^p(E)$ for $1 \leq p < \infty$	102
11 Compactness of A_ϕ in $L^\infty(E)$	102
Problems and Complements	104
2c On the Integral Defining the Double-Layer Potential	104
5c The Dirichlet Problem by Integral Equations	105
6c The Neumann Problem by Integral Equations	106

7c	Green's Function for the Neumann Problem	106
7.1c	Constructing $\mathcal{G}(\cdot; \cdot)$ for a Ball in \mathbb{R}^2 and \mathbb{R}^3	106
8c	Eigenvalue Problems	107
4	Integral Equations and Eigenvalue Problems	109
1	Kernels in $L^2(E)$	109
1.1	Examples of Kernels in $L^2(E)$	110
2	Integral Equations in $L^2(E)$	111
2.1	Existence of Solutions for Small $ \lambda $	111
3	Separable Kernels	112
3.1	Solving the Homogeneous Equations	113
3.2	Solving the Inhomogeneous Equation	113
4	Small Perturbations of Separable Kernels	114
4.1	Existence and Uniqueness of Solutions	115
5	Almost Separable Kernels and Compactness	116
5.1	Solving Integral Equations for Almost Separable Kernels	117
5.2	Potential Kernels Are Almost Separable	117
6	Applications to the Neumann Problem	118
7	The Eigenvalue Problem	119
8	Finding a First Eigenvalue and Its Eigenfunctions	121
9	The Sequence of Eigenvalues	122
9.1	An Alternative Construction Procedure of the Sequence of Eigenvalues	123
10	Questions of Completeness and the Hilbert–Schmidt Theorem	124
10.1	The Case of $K(x; \cdot) \in L^2(E)$ Uniformly in x	125
11	The Eigenvalue Problem for the Laplacean	126
11.1	An Expansion of Green's Function	127
	Problems and Complements	128
2c	Integral Equations	128
2.1c	Integral Equations of the First Kind	128
2.2c	Abel Equations ([2, 3])	128
2.3c	Solving Abel Integral Equations	129
2.4c	The Cycloid ([3])	130
2.5c	Volterra Integral Equations ([158, 159])	130
3c	Separable Kernels	131
3.1c	Hammerstein Integral Equations ([64])	131
6c	Applications to the Neumann Problem	132
9c	The Sequence of Eigenvalues	132
10c	Questions of Completeness	132
10.1c	Periodic Functions in \mathbb{R}^N	133
10.2c	The Poisson Equation with Periodic Boundary Conditions	134
11c	The Eigenvalue Problem for the Laplacian	134

5	The Heat Equation	135
1	Preliminaries	135
1.1	The Dirichlet Problem	136
1.2	The Neumann Problem	136
1.3	The Characteristic Cauchy Problem	136
2	The Cauchy Problem by Similarity Solutions	136
2.1	The Backward Cauchy Problem	140
3	The Maximum Principle and Uniqueness (Bounded Domains)	140
3.1	A Priori Estimates	141
3.2	Ill-Posed Problems	141
3.3	Uniqueness (Bounded Domains)	142
4	The Maximum Principle in \mathbb{R}^N	142
4.1	A Priori Estimates	144
4.2	About the Growth Conditions (4.3) and (4.4)	145
5	Uniqueness of Solutions to the Cauchy Problem	145
5.1	A Counterexample of Tychonov ([155])	145
6	Initial Data in $L^1_{loc}(\mathbb{R}^N)$	147
6.1	Initial Data in the Sense of $L^1_{loc}(\mathbb{R}^N)$	149
7	Remarks on the Cauchy Problem	149
7.1	About Regularity	149
7.2	Instability of the Backward Problem	150
8	Estimates Near $t = 0$	151
9	The Inhomogeneous Cauchy Problem	152
10	Problems in Bounded Domains	154
10.1	The Strong Solution	155
10.2	The Weak Solution and Energy Inequalities	156
11	Energy and Logarithmic Convexity	157
11.1	Uniqueness for Some Ill-Posed Problems	158
12	Local Solutions	158
12.1	Variable Cylinders	162
12.2	The Case $ \alpha = 0$	162
13	The Harnack Inequality	163
13.1	Compactly Supported Sub-Solutions	164
13.2	Proof of Theorem 13.1	165
14	Positive Solutions in S_T	167
14.1	Non-Negative Solutions	169
	Problems and Complements	171
2c	Similarity Methods	171
2.1c	The Heat Kernel Has Unit Mass	171
2.2c	The Porous Media Equation	172
2.3c	The p -Laplacean Equation	172
2.4c	The Error Function	173
2.5c	The Appell Transformation ([7])	173
2.6c	The Heat Kernel by Fourier Transform	173

2.7c	Rapidly Decreasing Functions	174
2.8c	The Fourier Transform of the Heat Kernel	174
2.9c	The Inversion Formula	175
3c	The Maximum Principle in Bounded Domains	176
3.1c	The Blow-Up Phenomenon for Super-Linear Equations	177
3.2c	The Maximum Principle for General Parabolic Equations	178
4c	The Maximum Principle in \mathbb{R}^N	178
4.1c	A Counterexample of the Tychonov Type	180
7c	Remarks on the Cauchy Problem	180
12c	On the Local Behavior of Solutions	180
6	The Wave Equation	183
1	The One-Dimensional Wave Equation	183
1.1	A Property of Solutions	184
2	The Cauchy Problem	185
3	Inhomogeneous Problems	186
4	A Boundary Value Problem (Vibrating String)	188
4.1	Separation of Variables	189
4.2	Odd Reflection	190
4.3	Energy and Uniqueness	190
4.4	Inhomogeneous Problems	191
5	The Initial Value Problem in N Dimensions	191
5.1	Spherical Means	192
5.2	The Darboux Formula	192
5.3	An Equivalent Formulation of the Cauchy Problem	193
6	The Cauchy Problem in \mathbb{R}^3	193
7	The Cauchy Problem in \mathbb{R}^2	196
8	The Inhomogeneous Cauchy Problem	198
9	The Cauchy Problem for Inhomogeneous Surfaces	199
9.1	Reduction to Homogeneous Data on $t = \Phi$	200
9.2	The Problem with Homogeneous Data	200
10	Solutions in Half-Space. The Reflection Technique	201
10.1	An Auxiliary Problem	202
10.2	Homogeneous Data on the Hyperplane $x_3 = 0$	202
11	A Boundary Value Problem	203
12	Hyperbolic Equations in Two Variables	204
13	The Characteristic Goursat Problem	205
13.1	Proof of Theorem 13.1: Existence	205
13.2	Proof of Theorem 13.1: Uniqueness	207
13.3	Goursat Problems in Rectangles	207
14	The Non-Characteristic Cauchy Problem and the Riemann Function	208
15	Symmetry of the Riemann Function	210

Problems and Complements	211
2c The d'Alembert Formula	211
3c Inhomogeneous Problems	211
3.1c The Duhamel Principle ([38])	211
4c Solutions for the Vibrating String	212
6c Cauchy Problems in \mathbb{R}^3	214
6.1c Asymptotic Behavior	214
6.2c Radial Solutions	214
6.3c Solving the Cauchy Problem by Fourier Transform	216
7c Cauchy Problems in \mathbb{R}^2 and the Method of Descent	217
7.1c The Cauchy Problem for $N = 4, 5$	218
8c Inhomogeneous Cauchy Problems	218
8.1c The Wave Equation for the N and $(N + 1)$ -Laplacian ..	218
8.2c Miscellaneous Problems	219
10c The Reflection Technique	221
11c Problems in Bounded Domains	221
11.1c Uniqueness	221
11.2c Separation of Variables	222
12c Hyperbolic Equations in Two Variables	222
12.1c The General Telegraph Equation	222
14c Goursat Problems	223
14.1c The Riemann Function and the Fundamental Solution of the Heat Equation	223
7 Quasi-Linear Equations of First-Order	225
1 Quasi-Linear Equations	225
2 The Cauchy Problem	226
2.1 The Case of Two Independent Variables	226
2.2 The Case of N Independent Variables	227
3 Solving the Cauchy Problem	227
3.1 Constant Coefficients	228
3.2 Solutions in Implicit Form	229
4 Equations in Divergence Form and Weak Solutions	230
4.1 Surfaces of Discontinuity	231
4.2 The Shock Line	231
5 The Initial Value Problem	232
5.1 Conservation Laws	233
6 Conservation Laws in One Space Dimension	234
6.1 Weak Solutions and Shocks	235
6.2 Lack of Uniqueness	236
7 Hopf Solution of The Burgers Equation	236
8 Weak Solutions to (6.4) When $a(\cdot)$ is Strictly Increasing	238
8.1 Lax Variational Solution	239
9 Constructing Variational Solutions I	240
9.1 Proof of Lemma 9.1	241

10	Constructing Variational Solutions II	242
11	The Theorems of Existence and Stability	244
11.1	Existence of Variational Solutions	244
11.2	Stability of Variational Solutions	245
12	Proof of Theorem 11.1	246
12.1	The Representation Formula (11.4)	246
12.2	Initial Datum in the Sense of $L^1_{\text{loc}}(\mathbb{R})$	247
12.3	Weak Forms of the PDE	248
13	The Entropy Condition	248
13.1	Entropy Solutions	249
13.2	Variational Solutions of (6.4) are Entropy Solutions	249
13.3	Remarks on the Shock and the Entropy Conditions	251
14	The Kruzhkov Uniqueness Theorem	253
14.1	Proof of the Uniqueness Theorem I	253
14.2	Proof of the Uniqueness Theorem II	254
14.3	Stability in $L^1(\mathbb{R}^N)$	256
15	The Maximum Principle for Entropy Solutions	256
	Problems and Complements	257
3c	Solving the Cauchy Problem	257
6c	Explicit Solutions to the Burgers Equation	259
6.2c	Invariance of Burgers Equations by Some Transformation of Variables	259
6.3c	The Generalized Riemann Problem	260
13c	The Entropy Condition	261
14c	The Kruzhkov Uniqueness Theorem	262
8	Non-Linear Equations of First-Order	265
1	Integral Surfaces and Monge's Cones	265
1.1	Constructing Monge's Cones	266
1.2	The Symmetric Equation of Monge's Cones	266
2	Characteristic Curves and Characteristic Strips	267
2.1	Characteristic Strips	268
3	The Cauchy Problem	269
3.1	Identifying the Initial Data $p(0, s)$	269
3.2	Constructing the Characteristic Strips	270
4	Solving the Cauchy Problem	270
4.1	Verifying (4.3)	271
4.2	A Quasi-Linear Example in \mathbb{R}^2	272
5	The Cauchy Problem for the Equation of Geometrical Optics	273
5.1	Wave Fronts, Light Rays, Local Solutions, and Caustics	274
6	The Initial Value Problem for Hamilton–Jacobi Equations	274
7	The Cauchy Problem in Terms of the Lagrangian	276

8	The Hopf Variational Solution	277
8.1	The First Hopf Variational Formula	278
8.2	The Second Hopf Variational Formula	278
9	Semigroup Property of Hopf Variational Solutions	279
10	Regularity of Hopf Variational Solutions	280
11	Hopf Variational Solutions (8.3) are Weak Solutions of the Cauchy Problem (6.4)	281
12	Some Examples	283
12.1	Example I	283
12.2	Example II	284
12.3	Example III	284
13	Uniqueness	285
14	More on Uniqueness and Stability	287
14.1	Stability in $L^p(\mathbb{R}^N)$ for All $p \geq 1$	287
14.2	Comparison Principle	288
15	Semi-Concave Solutions of the Cauchy Problem	288
15.1	Uniqueness of Semi-Concave Solutions	288
16	A Weak Notion of Semi-Concavity	289
17	Semi-Concavity of Hopf Variational Solutions	290
17.1	Weak Semi-Concavity of Hopf Variational Solutions Induced by the Initial Datum u_o	290
17.2	Strictly Convex Hamiltonian	291
18	Uniqueness of Weakly Semi-Concave Variational Hopf Solutions	293
9	Linear Elliptic Equations with Measurable Coefficients	297
1	Weak Formulations and Weak Derivatives	297
1.1	Weak Derivatives	298
2	Embeddings of $W^{1,p}(E)$	299
2.1	Compact Embeddings of $W^{1,p}(E)$	300
3	Multiplicative Embeddings of $W_o^{1,p}(E)$ and $\tilde{W}^{1,p}(E)$	300
3.1	Some Consequences of the Multiplicative Embedding Inequalities	301
4	The Homogeneous Dirichlet Problem	302
5	Solving the Homogeneous Dirichlet Problem (4.1) by the Riesz Representation Theorem	302
6	Solving the Homogeneous Dirichlet Problem (4.1) by Variational Methods	303
6.1	The Case $N = 2$	304
6.2	Gâteaux Derivative and The Euler Equation of $J(\cdot)$	305
7	Solving the Homogeneous Dirichlet Problem (4.1) by Galerkin Approximations	305
7.1	On the Selection of an Orthonormal System in $W_o^{1,2}(E)$	306

7.2	Conditions on \mathbf{f} and f for the Solvability of the Dirichlet Problem (4.1)	307
8	Traces on ∂E of Functions in $W^{1,p}(E)$	307
8.1	The Segment Property	307
8.2	Defining Traces	308
8.3	Characterizing the Traces on ∂E of Functions in $W^{1,p}(E)$	309
9	The Inhomogeneous Dirichlet Problem	309
10	The Neumann Problem	310
10.1	A Variant of (10.1)	311
11	The Eigenvalue Problem	312
12	Constructing the Eigenvalues of (11.1)	313
13	The Sequence of Eigenvalues and Eigenfunctions	315
14	A Priori $L^\infty(E)$ Estimates for Solutions of the Dirichlet Problem (9.1)	317
15	Proof of Propositions 14.1–14.2	318
15.1	An Auxiliary Lemma on Fast Geometric Convergence	319
15.2	Proof of Proposition 14.1 for $N > 2$	319
15.3	Proof of Proposition 14.1 for $N = 2$	320
16	A Priori $L^\infty(E)$ Estimates for Solutions of the Neumann Problem (10.1)	320
17	Proof of Propositions 16.1–16.2	322
17.1	Proof of Proposition 16.1 for $N > 2$	324
17.2	Proof of Proposition 16.1 for $N = 2$	325
18	Miscellaneous Remarks on Further Regularity	325
	Problems and Complements	326
1c	Weak Formulations and Weak Derivatives	326
1.1c	The Chain Rule in $W^{1,p}(E)$	326
2c	Embeddings of $W^{1,p}(E)$	327
2.1c	Proof of (2.4)	327
2.2c	Compact Embeddings of $W^{1,p}(E)$	328
3c	Multiplicative Embeddings of $W^{1,p}_o(E)$ and $\tilde{W}^{1,p}(E)$	329
3.1c	Proof of Theorem 3.1 for $1 \leq p < N$	329
3.2c	Proof of Theorem 3.1 for $p \geq N > 1$	331
3.3c	Proof of Theorem 3.2 for $1 \leq p < N$ and E Convex	332
5c	Solving the Homogeneous Dirichlet Problem (4.1) by the Riesz Representation Theorem	333
6c	Solving the Homogeneous Dirichlet Problem (4.1) by Variational Methods	334
6.1c	More General Variational Problems	334
6.8c	Gâteaux Derivatives, Euler Equations, and Quasi-Linear Elliptic Equations	336

8c	Traces on ∂E of Functions in $W^{1,p}(E)$	337
8.1c	Extending Functions in $W^{1,p}(E)$	337
8.2c	The Trace Inequality	338
8.3c	Characterizing the Traces on ∂E of Functions in $W^{1,p}(E)$	339
9c	The Inhomogeneous Dirichlet Problem	341
9.1c	The Lebesgue Spike	341
9.2c	Variational Integrals and Quasi-Linear Equations	341
10c	The Neumann Problem	342
11c	The Eigenvalue Problem	343
12c	Constructing the Eigenvalues	343
13c	The Sequence of Eigenvalues and Eigenfunctions	343
14c	A Priori $L^\infty(E)$ Estimates for Solutions of the Dirichlet Problem (9.1)	343
15c	A Priori $L^\infty(E)$ Estimates for Solutions of the Neumann Problem (10.1)	344
15.1c	Back to the Quasi-Linear Dirichlet Problem (9.1c)	344
10	DeGiorgi Classes	347
1	Quasi-Linear Equations and DeGiorgi Classes	347
1.1	DeGiorgi Classes	349
2	Local Boundedness of Functions in the DeGiorgi Classes	350
2.1	Proof of Theorem 2.1 for $1 < p < N$	351
2.2	Proof of Theorem 2.1 for $p = N$	352
3	Hölder Continuity of Functions in the DG Classes	353
3.1	On the Proof of Theorem 3.1	354
4	Estimating the Values of u by the Measure of the Set where u is Either Near μ^+ or Near μ^-	354
5	Reducing the Measure of the Set where u is Either Near μ^+ or Near μ^-	355
5.1	The Discrete Isoperimetric Inequality	356
5.2	Proof of Proposition 5.1	357
6	Proof of Theorem 3.1	358
7	Boundary DeGiorgi Classes: Dirichlet Data	359
7.1	Continuity up to ∂E of Functions in the Boundary DG Classes (Dirichlet Data)	360
8	Boundary DeGiorgi Classes: Neumann Data	361
8.1	Continuity up to ∂E of Functions in the Boundary DG Classes (Neumann Data)	363
9	The Harnack Inequality	364
9.1	Proof of Theorem 9.1 (Preliminaries)	364
9.2	Proof of Theorem 9.1. Expansion of Positivity	365
9.3	Proof of Theorem 9.1	365
10	Harnack Inequality and Hölder Continuity	367

11	Local Clustering of the Positivity Set of Functions in $W^{1,1}(E)$	368
12	A Proof of the Harnack Inequality Independent of Hölder Continuity	370
References		373
Index		381