
Contents

Part I Classical Geometry

| | | |
|----------|--|-----------|
| 1 | Thales and Pythagoras | 3 |
| 1.1 | Thales' Theorem | 3 |
| 1.2 | Similar Figures | 5 |
| 1.3 | Properties of Angles | 7 |
| 1.4 | The Regular Pentagon | 9 |
| 1.5 | The Computation of Areas | 11 |
| 1.6 | A Remarkable Babylonian Document | 13 |
| 1.7 | The Pythagorean Theorem | 14 |
| 1.8 | Three Famous Problems of Greek Geometry | 19 |
| 1.9 | Exercises | 21 |
| 2 | The Elements of Euclid | 27 |
| 2.1 | Book I | 29 |
| 2.2 | Book III. Properties of Circles and Angles | 39 |
| 2.3 | Books V and VI. Real Numbers and Thales' Theorem | 42 |
| 2.4 | Books VII and IX. Number Theory | 43 |
| 2.5 | Book XI. Spatial Geometry and Solids | 45 |
| 2.6 | Book XII. Areas and Volumes of Circles, Pyramids, Cones and Spheres | 48 |
| 2.7 | Epilogue | 52 |
| 2.8 | Exercises | 54 |
| 3 | Conic Sections | 61 |
| 3.1 | The Parabola | 62 |
| 3.2 | The Ellipse | 64 |
| 3.3 | The Hyperbola | 70 |
| 3.4 | The Area of a Parabola | 72 |
| 3.5 | Exercises | 73 |

| | | |
|----------|---|-----|
| 4 | Further Results in Euclidean Geometry | 79 |
| 4.1 | The Conchoid of Nicomedes, the Trisection of an Angle | 79 |
| 4.2 | The Archimedean Spiral | 81 |
| 4.3 | The Four Classical Centres of the Triangle | 82 |
| 4.4 | The Theorems of Menelaus and Ceva | 87 |
| 4.5 | The Theorems of Apollonius–Pappus–Stewart | 89 |
| 4.6 | The Euler Line and the Nine-Point Circle | 91 |
| 4.7 | Excircles and the Nagel Point | 93 |
| 4.8 | Miquel's Theorems | 94 |
| 4.9 | Steiner's Circle Theorems | 98 |
| 4.10 | Morley's Theorem | 104 |
| 4.11 | Exercises | 106 |
| 5 | Trigonometry | 113 |
| 5.1 | Ptolemy and the Chord Function | 113 |
| 5.2 | Regiomontanus and Euler's Trigonometric Functions | 116 |
| 5.3 | Arbitrary Triangles | 119 |
| 5.4 | Trigonometric Solution of Malfatti's Problem | 121 |
| 5.5 | The Stereographic Projection | 124 |
| 5.6 | The Spherical Trigonometry of Right-Angled Triangles | 127 |
| 5.7 | The Spherical Trigonometry of General Triangles | 133 |
| 5.8 | The Area of a Spherical Triangle | 139 |
| 5.9 | Trigonometric Formulas for the Conics | 140 |
| 5.10 | The Great Discoveries of Kepler and Newton | 141 |
| 5.11 | Exercises | 150 |

Part II Analytic Geometry

| | | |
|----------|--|-----|
| 6 | Descartes' Geometry | 159 |
| 6.1 | The Principles of Descartes' Geometry | 159 |
| 6.2 | The Regular Heptagon and Enneagon | 162 |
| 6.3 | The Trisection of an Angle and Cubic Equations | 164 |
| 6.4 | Regular Polygons in the Unit Circle | 166 |
| 6.5 | Van Roomen's Famous Challenge | 168 |
| 6.6 | A Geometric Theorem of Fermat | 170 |
| 6.7 | Heron's Formula for the Area of a Triangle | 171 |
| 6.8 | The Euler–Brahmagupta Formula for a Cyclic Quadrilateral | 174 |
| 6.9 | The Cramer–Castillon Problem | 175 |
| 6.10 | Exercises | 178 |
| 7 | Cartesian Coordinates | 185 |
| 7.1 | Equations of Lines and Circles | 185 |
| 7.2 | The Problem of Pappus | 188 |
| 7.3 | Conic Sections: Poles, Polars and Tangents | 189 |

| | | |
|-----------|---|------------|
| 7.4 | Problems of Minimum and Maximum | 193 |
| 7.5 | Some Famous Curves and Their Tangents | 199 |
| 7.6 | Curvature | 210 |
| 7.7 | The Euler Line by Euler | 214 |
| 7.8 | The Simson Line and Sturm's Circles | 217 |
| 7.9 | The Erdős–Mordell Inequality and the Steiner–Lehmus Theorem | 222 |
| 7.10 | The Butterfly | 225 |
| 7.11 | Thébault's Theorem | 227 |
| 7.12 | Billiards in an Ellipse | 230 |
| 7.13 | Urquhart's 'Most Elementary Theorem of Euclidean Geometry' | 231 |
| 7.14 | Exercises | 232 |
| 8 | To be Constructible, or not to be | 241 |
| 8.1 | The Complex Plane and the Logarithmic Spiral | 242 |
| 8.2 | Constructions with Ruler and Compass | 246 |
| 8.3 | The Method of Gauss and Vandermonde | 247 |
| 8.4 | The Regular 17-Sided Polygon | 249 |
| 8.5 | Constructions Impossible with Ruler and Compass | 251 |
| 8.6 | Exercises | 254 |
| 9 | Spatial Geometry and Vector Algebra | 259 |
| 9.1 | First Applications of Vectors | 264 |
| 9.2 | Gaussian Elimination, Volume and Determinant | 265 |
| 9.3 | Norm and Scalar Product | 268 |
| 9.4 | The Outer Product | 270 |
| 9.5 | Spherical Trigonometry Revisited | 274 |
| 9.6 | Pick's Theorem | 277 |
| 9.7 | A Theorem on Pentagons in Space | 280 |
| 9.8 | Archimedean Solids | 282 |
| 9.9 | Exercises | 287 |
| 10 | Matrices and Linear Mappings | 291 |
| 10.1 | Changes of Coordinates | 291 |
| 10.2 | Linear Mappings | 292 |
| 10.3 | Gram's Determinant | 296 |
| 10.4 | Orthogonal Mappings and Isometries | 300 |
| 10.5 | Skew-Symmetric Matrices, the Cayley Transform | 304 |
| 10.6 | Eigenvalues and Eigenvectors | 307 |
| 10.7 | Quadratic Forms | 308 |
| 10.8 | Exercises | 313 |

| | |
|--|-----|
| 11 Projective Geometry | 319 |
| 11.1 Perspective and Central Projection | 320 |
| 11.2 Poncelet's Principle of Central Projection..... | 323 |
| 11.3 The Projective Line..... | 330 |
| 11.4 The Projective Plane | 335 |
| 11.5 The Principle of Duality | 338 |
| 11.6 The Projective Theory of Conics | 340 |
| 11.7 Exercises | 342 |
| 12 Solutions to the Exercises | 345 |
| References | 403 |
| Figure Sources and Copyright | 419 |